The rank one abelian Stark conjecture

Berkeley-Stanford Number Theory Seminar

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The purpose of this note is to collect some basic facts about the rank one abelian Stark conjecture, following Section 1 of [DG11], for the Berkeley-Stanford Number Theory Seminar. Another nice reference (in French) is Tate's book [Tat84, Chapitre IV]. All errors and pedantry are due to me.

1 Introduction and Motivation

The notation is mostly as usual: K will be a number field, d its discriminant, h its class number, r_1 and r_2 the number of real embeddings and pairs of complex conjugate embeddings of K respectively, and $r := r_1 + r_2 - 1$. We write M_K for the set of places of K, and $S \subseteq M_K$ will be a finite subset of places, always containing at least the infinite ones. We also write e for the number of roots of unity lying in K, and R for the regulator[1] of K. If we need to distinguish between multiple number fields then these letters will get the appropriate subscripts.

We begin by recalling the class number formula:

Theorem 1.1 (Class number formula)**.** [Coh07, Theorem 10.5.1] Let $\zeta_K(s)$ be the zeta function of K, that is,

$$\zeta_K(s) := \sum_{\mathfrak{n} \subseteq \mathcal{O}_K} N(\mathfrak{n})^{-s}$$

defined first for $\Re(s) > 1$, and then extended meromorphically to $\mathbf{C}$. It is holomorphic everywhere except for a simple pole at $s = 1$. Then

$$\mathrm{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} hR}{e\sqrt{|d|}}.$$

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[1]Recall that this is the determinant of any rank-r minor of the $r \times (r+1)$ matrix $[\log(|\sigma_j(e_i)|^{n_j})]_{1 \le i \le r, 1 \le j \le r+1}$, where the e_i are a system of fundamental units of K, the σ_j are the embeddings $K \hookrightarrow \mathbf{C}$ with complex conjugates identified, and $n_j = 1$ if σ_j is real and 2 if it is complex.

The rank one abelian Stark conjecture is a generalization of this. There are two things that we would like to do. First, we allow some dependency on a finite set of places $S \subseteq M_K$. Second, we allow a relative extension of number fields.

To make our generalization, we slightly reformulate the statement of the class number formula.

Proposition 1.2 (Functional equation)**.** Let

$$\Lambda_K(s) := |d|^{s/2}\gamma(s)^{r_1+r_2}\gamma(s+1)^{r_2}\zeta_K(s),$$

where $\gamma(s) := \pi^{-s/2}\Gamma(s/2)$. Then $\Lambda_K(s) = \Lambda_K(1-s)$.

Plug in $s = 0$, and get

$$\zeta_K(0) = \frac{|d|^{1/2}\gamma(2)^{r_2}\zeta_K(1)}{\gamma(0)^{r_1+r_2}}.$$

The numerator has a pole of order 1 from the $\zeta_K(1)$ term, and the denominator has a pole of order $r_1 + r_2$, so ζ_K has a zero of order $r = r_1 + r_2 - 1$ at 0. We now compute the leading nonzero coefficient in the Taylor expansion of ζ_K at 0, which is

$$\lim_{s\to 0} s^{-r}\zeta_K(0) = \lim_{s\to 0} \frac{|d|^{1/2}\gamma(2)^{r_2}\zeta_K(1-s)}{s^r\gamma(s)^{r_1+r_2}} = \lim_{s\to 0} \frac{|d|^{1/2}\pi^{-r_2}s\zeta_K(1-s)}{2^{r+1}(s/2)^{r+1}\pi^{-s(r_1+r_2)/2}\Gamma(s)^{r_1+r_2}} = -\frac{hR}{e}.$$

To generalize, we define the *S-imprimitive Dedekind zeta function*, for $S \subseteq M_K$ as above, as

$$\zeta_{K,S}(s) := \sum_{\mathfrak{n}\subseteq\mathcal{O}_K,(\mathfrak{n},S)=1} N(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}\notin S}(1 - N(\mathfrak{p})^{-s})^{-1}$$

for $\Re(s) > 1$, and then extended to all of $\mathbf{C}$ (via an analogous functional equation) to a function that is holomorphic everywhere except for a simple pole at 1. Moreover, we have a class-number formula analogous to the above:

Theorem 1.3 (S-class number formula)**.** [Das99, Corollary 3.1.4] $\zeta_{K,S}$ vanishes to order $|S| - 1$ at 0, and

$$\lim_{s\to 0} s^{-(|S|-1)}\zeta_{K,S}(s) = -\frac{h_S R_S}{e}.$$

Here, $h_S := |\mathrm{Cl}(\mathcal{O}_{K,S})|$, and R_S is the S-regulator of K, which is defined in an analogous fashion to the usual regulator (in the case where S contains only the infinite places) via Dirichlet's S-unit theorem.

Now we allow for a *abelian* extension K/F of number fields with Galois group $G := \mathrm{Gal}(K/F)$. We need to set up some notation. Let $S \subseteq M_F$ be a finite subset of places containing the infinite ones as well as the ones that ramify. If $\mathfrak{n} \subseteq \mathcal{O}_F$ is a nonzero ideal not divisible by a prime ramifying in K, then we denote by $\sigma_{\mathfrak{n}} \in G$ the associated Frobenius element of $\mathfrak{n}$ (i.e. the image of $\mathfrak{n}$ under the Artin map). For each element $\sigma \in G$, we have a partial zeta function

$$\zeta_{K/F,S}(\sigma, s) := \sum_{\mathfrak{n} \subseteq \mathcal{O}_F, (\mathfrak{n}, S)=1, \sigma_{\mathfrak{n}}=\sigma} N(\mathfrak{n})^{-s}$$

originally defined for $\Re(s) > 1$, and then extended to all of $\mathbf{C}$ to a function that is holomorphic everywhere except for a simple pole at 1. Note that when $K = F$, we get $\zeta_{F,S}$.

We might also instead like to express things in terms of L-functions. For a character $\chi : G \to \mathbf{C}^\times$, we define

$$L_S(\chi, s) := \sum_{\sigma \in G} \chi(\sigma) \zeta_{K/F,S}(\sigma, s).$$

It enjoys a functional equation and moreover an Euler product valid for $\Re(s) > 1$ ([Neu99, Proposition VII.8.1] essentially gives the derivation):

$$L_S(\chi, s) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\sigma_{\mathfrak{p}}) N(\mathfrak{p})^{-s})^{-1}.$$

Note that this gives $L_S(\mathbf{1}, s) = \zeta_{F,S}(s)$ for the trivial character $\mathbf{1}$. It is also not hard to see that

$$\zeta_{K,S_K}(s) = \prod_{\chi \in \widehat{G}} L_S(\chi, s), \tag{1.1}$$

where S_K is the set of places of K lying above the places of S. Indeed, using the fact that $\prod_{\chi \in \widehat{H}} (1 - \chi(h)x) = (1 - x^m)^{n/m}$ for any finite abelian group H of order n and any element $h \in H$ of order m, we have for $\mathfrak{p} \notin S$:

$$\prod_{\chi \in \widehat{G}} (1 - \chi(\sigma_{\mathfrak{p}}) N(\mathfrak{p})^{-s})^{-1} = (1 - N(\mathfrak{p})^{-sf_{\mathfrak{p}}})^{-[K:F]/f_{\mathfrak{p}}}$$

and so using $\mathfrak{P}_{\mathfrak{p}}$ to denote some prime of K lying over $\mathfrak{p}$,

$$\prod_{\chi \in \widehat{G}} L_S(\chi, s) = \prod_{\mathfrak{p} \notin S} (1 - N(\mathfrak{p})^{-sf_{\mathfrak{p}}})^{-[K:F]/f_{\mathfrak{p}}} = \prod_{\mathfrak{p} \notin S} \left(1 - N(\mathfrak{P}_{\mathfrak{p}})^{-s}\right)^{-[K:F]/f_{\mathfrak{p}}} = \prod_{\mathfrak{P} \notin S_K} \left(1 - N(\mathfrak{P})^{-s}\right)^{-1}$$

as there are $[K : F]/f_{\mathfrak{p}}$ primes lying over the unramified $\mathfrak{p}$.

The vague idea of Stark's conjecture is that via the Equation 1.1, the leading coefficient $-h_{S_K} R_{S_K}/e_K$ of the expansion of ζ_{K,S_K} at 0 should factor over the L-functions $L_S(\chi, s)$. In other words, the leading coefficients of the $L_S(\chi, s)$ should at the very least be a rational number times some determinant in logarithms of (absolute values of) algebraic numbers (where these algebraic numbers should have something to do with χ, of course). Certainly we will need some other assumptions; these are discussed in the next section.

2 Statement of the conjecture

We may now state the rank-one abelian Stark conjecture, which is formulated in [Sta75, pg. 61]. We set up some notation first. Let K/F be as above, and let $S \subseteq M_F$ be a finite subset of places containing the infinite ones *and the ones that ramify in* K. Suppose S contains at least one place $v \in M_F$ that splits completely in K and that $|S| \geq 2$. Let $U_{v,S} = U_{v,S}(K)$ be the set of all elements in $K^\times$ such that:

- If $|S| \geq 3$, then $|u|_{w'} = 1$ for all $w' \nmid v$. In other words $U_{v,S}$ consists of the $\{v\}$-units of K.
- If $|S| = \{v, v'\}$, then $|u|_{w'}$ is constant over all w' above v', and $|u|_{w'} = 1$ for all $w' \notin S_K$. In other words $U_{v,S}$ consists of the S-units of K that also satisfy the second "constancy" condition.

Finally, the statement of the conjecture is:

Conjecture 2.1 (Rank one abelian Stark conjecture)**.** Fix a place $w \in M_K$ above v (recall $v \in S$ and v splits completely in K). Then there exists $u \in U_{v,S}$ such that

$$\zeta'_{K/F,S}(\sigma, 0) = -\frac{1}{e_K} \log|\sigma(u)|_w \tag{2.1}$$

for all $\sigma \in G$, and such that $K(u^{1/e_K})/F$ is an abelian extension.

We now make some few basic remarks.

Remark 2.2. The choice of w does not matter. Moreover, if u exists, it is unique up to roots of unity. Indeed, the conditions on $U_{v,S}$ and the desired equality imply that the valuations of u at all places are fixed (when $|S| = 2$, this is true by the product formula and the fact that G acts transitively on places).

Remark 2.3. The condition 2.1 is equivalent to

$$L'_S(\chi, 0) = -\frac{1}{e_K} \sum_{\sigma \in G} \chi(\sigma) \log|\sigma(u)|_w \tag{2.2}$$

for all characters χ of G. This is obvious from

$$L'_S(\chi, 0) = \sum_{\sigma \in G} \chi(\sigma) \zeta'_{K/F,S}(\sigma, 0).$$

Proposition 2.4. If S contains at least two places that split completely in K, then Conjecture 2.1 is true.

For the proof, we will need a result which comes more from representation theory:

Lemma 2.5. The order of vanishing $r_S(\chi)$ of $L_S(\chi, s)$ at 0 is $|\{v \in S : \chi(G_v) = 1\}|$ if $\chi \neq 1$ (where G_v is the decomposition group over v), and $|S| - 1$ if $\chi = 1$.

Unfortunately we don't have time to prove this. For the proof, see [Das99, Proposition 3.2.4] or [Tat84, Proposition I.3.4]. But note that with our assumptions on S, this means that $L_S(\chi, s)$ always vanishes at 0, which might explain the strange-seeming assumptions. Also, by tweaking various assumptions, it is possible to get higher-rank (larger $r_S(\chi)$) analogues of the conjecture, particularly in the case when the extension K/F is no longer abelian and χ is no longer a 1-dimensional character of G.

Proof of Proposition 2.4. Recall that v splitting completely means that $G_v = 1$. Now, if $|S| \geq 3$, in all cases we have $L_S(\chi, s)$ vanishing to order at least 2 at 0, so using 2.2 we see that we may take $u = 1$. In the other case $|S| = 2$, let $S = \{v, v'\}$, so by the Dirichlet S-unit theorem the rank of $\mathcal{O}_{F,S}^\times$ is 1. By inverting if necessary, let $\epsilon_0 \in F^\times$ be a fundamental S-unit with $|\epsilon_0|_v > 1$. By Theorem 1.3, we have $\zeta'_{F,S}(0) = -h_{F,S} \log|\epsilon_0|_v / e_F$.

We claim that $[K : F]$ divides $h_{F,S}$. This is seen by class field theory: let $\mathfrak{m}$ be the modulus that is the product of the places in S. Since the S-class group $\mathrm{Cl}(\mathcal{O}_{F,S})$ is a quotient of the ray class group modulo $\mathfrak{m}$, we may find a finite abelian extension L/F such that $[L : F] = |\mathrm{Cl}(\mathcal{O}_{F,S})| = h_{F,S}$. The congruence subgroup of $I(\mathcal{O}_F)_\mathfrak{m}$ corresponding to this extension is just the preimage of the principal fractional ideals of $I(\mathcal{O}_{F,S})$, i.e. those of the form $I \cdot (x)$ for I a fractional ideal *coprime* to primes *not* in S and $x \in F^\times$. Now K/F is unramified everywhere by assumption, so consider the Artin map $\psi_{K/F} : I(\mathcal{O}_F)_\mathfrak{m} \subseteq I(\mathcal{O}_F) \to \mathrm{Gal}(K/F)$. It kills the principal fractional ideals of $I(\mathcal{O}_F)$ as well as the finite primes of S (which split completely), so $\psi_{K/F}$ contains the congruence subgroup defining L in its kernel. Hence $K \subseteq L$.

Going back to the main proof, we see that $\frac{e_K h_{F,S}}{e_F [K:F]}$ is an integer m, so set $\epsilon := \epsilon_0^m \in F$. In particular $|\epsilon|_{w'}$ is constant over all $w'|v'$. Moreover $K(\epsilon^{1/e_K})$ is a subfield of the composite of K/F and $F(\epsilon_0^{1/e_F})/F$, so it is abelian over F (note F contains e_F roots of unity so the latter is abelian). So it remains to check the condition 2.2. This is easy:

$$L'_S(\mathbf{1}, 0) = \zeta'_{F,S}(0) = -\frac{[K:F]}{e_K} \log|\epsilon|_v = -\frac{1}{e_K} \sum_{\sigma \in G} \mathbf{1}(\sigma) \log|\sigma(\epsilon)|_w$$

and for nontrivial χ,

$$-\frac{1}{e_K} \sum_{\sigma \in G} \chi(\sigma) \log|\sigma(\epsilon)|_w = -\frac{\log|\epsilon|_v}{e_K} \sum_{\sigma \in G} \chi(\sigma) = 0.$$

But $L'_S(\chi, 0) = 0$ by Lemma 2.5, because at least 2 places (v, v') are totally split in K. $\square$

Corollary 2.6. Conjecture 2.1 is true when $K = F$.

As another corollary of the above, because S must contain all the infinite places and complex places of F by definition split completely in K/F, if S contains at least two complex places then the Stark conjecture is true. Therefore the rank one abelian Stark conjecture can be split into three "nontrivial" cases:

1. F is totally real, v is infinite, and every other infinite place of F ramifies (so any infinite place of K not above v is complex).

2. F has one complex place v and all other infinite places are real and ramify in K. In particular K is totally imaginary.

3. F is totally real, v is finite, and K is totally complex.

This is the trichotomy mentioned in [DG11, Section 1.4].

Moreover, note that away from the trivial case of Proposition 2.4, the choice of $v \in S$ is "fixed," in that S can only contain one totally split place. In light of this, there is a more general representation-theoretic formulation of the Stark conjecture due to Tate that omits the choice of the place v; see [Das99, Chapter 6] or [Tat82].

Remark 2.7. There are also function field and p-adic analogues of Conjecture 2.1; see [Tat84, Chapitres V, VI].

To conclude this section it is worthwhile to mention two additional results which unfortunately we do not have time to prove.

Proposition 2.8. Suppose $S \subseteq M_F$ satisfies the hypotheses of the Stark conjecture, and Conjecture 2.1 is true for K/F and S, then for any $S \subseteq S' \subseteq M_K$, Conjecture 2.1 is also true for K/F and S'.

Proof. See [Das99, Proposition 4.3.7] or [Tat84, Proposition IV.3.4]. □

Proposition 2.9. Suppose $K/E/F$ is a tower of number fields with all extensions abelian, $S \subseteq M_F$ satisfies the hypotheses of the Stark conjecture for the extension K/F (so it automatically does for the extension E/F), and Conjecture 2.1 is true for the extension K/F and S. Then Conjecture 2.1 is true for the extension E/F and S.

Proof. See [Das99, Proposition 4.3.8] or [Tat84, Proposition IV.3.5]. □

3 The case $F = \mathbf{Q}$

We will now prove Stark's conjecture in the case when the ground field is $F = \mathbf{Q}$. This is originally done by Stark in [Sta75].

Theorem 3.1. Conjecture 2.1 is true with $F = \mathbf{Q}$.

Proof of Theorem 3.1, v infinite. We first treat the case where $v = \infty$ is the infinite place of $\mathbf{Q}$. Because v totally splits in $K/\mathbf{Q}$, which is abelian, K is contained in the maximal totally real subfield $\mathbf{Q}(\zeta_m)^+ \coloneqq \mathbf{Q}(\zeta_m + \zeta_m^{-1})$ of a cyclotomic field $\mathbf{Q}(\zeta_m)$ (where $m \geq 3$, m odd or divisible by 4), so we may reduce to this case by Proposition 2.9. Also, by Proposition 2.8, we only need to prove the case where S consists of $v = \infty$ and the primes dividing m, since those are the ramified primes in $K = \mathbf{Q}(\zeta_m)^+$. Let σ_a be the image of a under the natural map $(\mathbf{Z}/m\mathbf{Z})^\times \to G$, so that $\sigma_a = \sigma_{-a}$.

In this case we luckily can just write down the desired unit u; we claim

$$u \coloneqq 2 - \zeta_m - \zeta_m^{-1} = (\zeta_m - 1)(\zeta_m^{-1} - 1)$$

works (note that u is a totally positive element). In the case when m is not power of a prime, u is a unit in $\mathcal{O}_K^\times$ as $\zeta_m - 1$, $\zeta_m^{-1} - 1$ are units in $\mathbf{Z}[\zeta_m]^\times$ [Was96, Proposition 2.8]. Otherwise $m = p^k$, $S = \{\infty, p\}$, and $(\zeta_m - 1)^{(p-1)p^{k-1}} = p\mathbf{Z}[\zeta_m]$ as ideals of $\mathbf{Z}[\zeta_m]$, so that $|u|_w = 1$ for any place w not lying over p. Since there is only one place of K lying over p, we get $u \in U_{v,S}$ at least. Also, we have $e_K = 2$, and $u = -(\zeta_{2m} - \zeta_{2m}^{-1})^2$, so $K(u^{1/e_K}) \subseteq \mathbf{Q}(\zeta_{4m})$ is abelian over $\mathbf{Q}$.

So it remains to verify Equation 2.1 for a place w of K over ∞. One has for $0 < a < m$:

$$\zeta_{K/\mathbf{Q},S}(\sigma_a, s) = \sum_{n \in \mathbf{N} : n \bmod m \equiv a} n^{-s} + \sum_{n \in \mathbf{N} : n \bmod m \equiv -a} n^{-s} = m^{-s}\zeta\left(\frac{a}{m}, s\right) + m^{-s}\zeta\left(1 - \frac{a}{m}, s\right),$$

where $\zeta(b, s) = \sum_{n=0}^{\infty}(n + b)^{-s}$ for $b > 0$ is the Hurwitz zeta function (for background on this, including the analytic continuation to $s \in \mathbf{C} - \{1\}$, see [Coh07, Section 9.6]). Granting for the moment Lerch's formula

$$\zeta'(b, 0) \coloneqq \frac{\partial}{\partial s}\zeta(b, s)|_{s=0} = \log(\Gamma(b)) - \frac{\log(2\pi)}{2}, \tag{3.1}$$

we get

$$\begin{aligned}
\zeta'_{K/\mathbf{Q},S}(\sigma_a, 0) &= -\log(m)\left(\zeta\left(\frac{a}{m}, 0\right) + \zeta\left(1 - \frac{a}{m}, 0\right)\right) + \log\left(\Gamma\left(\frac{a}{m}\right)\Gamma\left(1 - \frac{a}{m}\right)\right) - \log(2\pi) \\
&= -\log\left(\frac{\sin(\pi a/m)}{\pi}\right) - \log(2\pi) \\
&= -\frac{1}{2}\log\left(2 - 2\cos\left(\frac{2\pi a}{m}\right)\right) \\
&= -\frac{1}{2}\log(\sigma_a(u)).
\end{aligned}$$

Here we also use the formula $\zeta(a, 1-n) = -B_n(a)/n$ valid for $n \in \mathbf{N}$, where $B_n(x)$ is the nth Bernoulli polynomial [Coh07, Corollary 9.6.10]. In particular $\zeta(a, 0) = 1/2 - a$. □

To finish the above proof:

Proof of Equation 3.1. For this we follow the argument of Berndt [Ber85]. We begin with the formula [Coh07, Proposition 9.6.7]

$$\sum_{m=0}^{N}(m+x)^s = \zeta(x,-s) + \frac{(N+x)^{s+1}}{s+1} + \frac{(N+x)^s}{2} + \sum_{j=2}^{k}\binom{s}{j-1}\frac{B_j}{j}(N+x)^{s-j+1} + R_k(s,x,N), \tag{3.2}$$

where

$$R_k(s,x,N) \coloneqq (-1)^k\binom{s}{k}\int_N^\infty (t+x)^{s-k}B_k(\{t\})dt, \quad \binom{s}{j} \coloneqq \frac{s(s-1)\cdots(s-j+1)}{j!}$$

and $\{t\}$ means the fractional part of t. This is valid for $s \in \mathbf{C} - \{1\}$, $x > 0$ real, $N \geq 0$, $k \geq \Re(s)+1, k \geq 1$,[2] and $B_j = B_j(0)$ is the jth Bernoulli number. The formula 3.2 is derived from the Euler-Maclaurin summation formula[3] (cf. [Coh07, Corollary 9.2.3, Proposition 9.2.5]). Plug in $N = 0$ and get

$$\zeta(x,-s) = -\frac{x^{s+1}}{s+1} - \sum_{j=1}^{k}\binom{s}{j-1}\frac{B_j}{j}x^{s-j+1} + (-1)^{k-1}\binom{s}{k}\int_0^\infty (t+x)^{s-k}B_k(\{t\})dt.$$

Now we take the derivative with respect to s and set $s = 0$. This gives

$$-\zeta'(x,0) = -x\log(x) + x + \frac{\log(x)}{2} + \sum_{j=2}^{k}(-1)^j\frac{B_j}{j(j-1)x^{j-1}} + \frac{1}{k}\int_0^\infty \frac{B_k(\{t\})}{(t+x)^k}dt$$

Set $k = 1$, and get

$$\zeta'(x,0) = x\log(x) - x - \frac{\log(x)}{2} - \int_0^\infty \frac{\{t\}-1/2}{t+x}dt. \tag{3.3}$$

Now

$$-\int_0^\infty \frac{\{t\}-1/2}{t+x}dt = \lim_{n\to\infty}\left(-\int_0^n \frac{t+x}{t+x}dt + \int_0^n \frac{1/2+x}{t+x}dt + \sum_{k=0}^{n-1}k\int_k^{k+1}\frac{1}{t+x}dt\right),$$

[2] One has to be slightly careful to extend this formula from $k > \Re(s)+1$ to $k = \Re(s)+1$ due to convergence issues of the integral, but we ignore this detail.

[3] Vaguely, approximating $\sum_{m=0}^{N}(m+x)^s$ via $\zeta(x,-s)$ and a corresponding integral from N to ∞.

so Equation 3.3 is

$$\zeta'(x,0) = \lim_{n\to\infty}\left(-x-n-\sum_{k=0}^{n}\log(k+x)+\left(n+x+\frac{1}{2}\right)\log(n+x)\right). \tag{3.4}$$

Using the well-known identity $\zeta'(0) = \log(2\pi)/2$ in the case $x = 1$, we see that

$$\begin{aligned}\zeta'(x,0)-\log(2\pi)/2 &= \lim_{n\to\infty}\left(1-x+\sum_{k=0}^{n}(\log(k+1)-\log(k+x))+\left(n+x+\frac{1}{2}\right)\log(n+x)-\left(n+\frac{3}{2}\right)\log(n+1)\right)\\ &= \lim_{n\to\infty}\left(1-x+\sum_{k=0}^{n}(\log(k+1)-\log(k+x))+\left(n+x+\frac{3}{2}\right)\log\left(\frac{n+x}{n+1}\right)-\log(n+x)+x\log(n+1)\right)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\log(k)+\log(n+1)-\sum_{k=0}^{n}\log(k+x)-\log(n+x)+x\log(n+1)\right)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\log(k)-\sum_{k=0}^{n}\log(k+x)+x\log(n+1)\right).\end{aligned} \tag{3.5}$$

Finally we recall the identity

$$\Gamma(x) = \lim_{n\to\infty}\frac{(n+1)^x n!}{x(x+1)\cdots(x+n)}$$

and so

$$\log(\Gamma(x)) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\log(k)+x\log(n+1)-\sum_{k=0}^{n}\log(k+x)\right) = \zeta'(x,0)-\frac{\log(2\pi)}{2}$$

as needed. □

It remains to take care of the case when v is a finite place of $\mathbf{Q}$.

Proof (sketch) of Theorem 3.1, v finite. As before we may assume that $K = \mathbf{Q}(\zeta_m)$ is cyclotomic, $v = p$, and $\mathfrak{p}$ is an ideal of K lying over p; note that $p \equiv 1 \bmod m$ and $N(\mathfrak{p}) = p$ by the totally split assumption. By Proposition 2.8 we may assume S consists of ∞, the primes l dividing m, and p, but this is not needed for the below discussion. Let $\chi_{\mathfrak{p}}$ be the (unique) group homomorphism $(\mathcal{O}_K/\mathfrak{p})^\times \to \mathcal{O}_K^\times$ (with image in μ_K) given by $\chi_{\mathfrak{p}}(x \bmod \mathfrak{p}) \equiv x^{\frac{p-1}{m}} \bmod \mathfrak{p}$ for $x \in \mathcal{O}_K$. Also, fix a nontrivial group homomorphism ψ from the additive group $\mathcal{O}_K/\mathfrak{p} \cong \mathbf{Z}/p\mathbf{Z}$ to μ_p. Define a Gauss sum

$$G(\chi_{\mathfrak{p}}) \coloneqq \sum_{x \bmod \mathfrak{p}} \chi_{\mathfrak{p}}(x)\psi(x) \in \mathbf{Q}(\zeta_{pm}),$$

where the sum runs over the residue classes in $\mathcal{O}_K/\mathfrak{p}$. Finally set

$$\beta(\mathfrak{p}) \coloneqq \left(\frac{G(\chi_{\mathfrak{p}})}{\sqrt{\pm p}}\right)^{e_K}, \tag{3.6}$$

where the sign is chosen so that $\sqrt{\pm p} \in \mathbf{Q}(\zeta_p)$. It turns out that $\beta(\mathfrak{p})$ does not depend on the choice of ψ.

It is now convenient to discuss a generalization of the Stark conjecture, called the *Brumer-Stark(-Tate) conjecture*. Put $T = S - \{v\}$, so that

$$\zeta_{K/\mathbf{Q},S}(\sigma, s) = \sum_{n\in\mathbf{Q},(n,T)=1,\sigma_n=\sigma} n^{-s} - \sum_{n\in\mathbf{Q},(n,T)=1,p|n,\sigma_n=\sigma} n^{-s} = (1-p^{-s})\zeta_{K/\mathbf{Q},T}(\sigma, s).$$

Taking derivatives gives $\zeta'_{K/\mathbf{Q},S}(\sigma, 0) = \log(p)\zeta_{K/\mathbf{Q},T}(\sigma, 0)$. Therefore *if* Condition 2.1 holds with our data, then $\theta_T \cdot \mathfrak{p}$ is a principal ideal generated by the relevant Stark unit u, where

$$\theta_T \coloneqq e_K \sum_{\sigma\in G} \zeta_{K/\mathbf{Q},T}(\sigma, 0)\sigma^{-1} \in \mathbf{Z}[G]. \tag{3.7}$$

Here $\mathbf{Z}[G]$ acts on fractional ideals of $\mathcal{O}_K$ by $\sum_\sigma n_\sigma \sigma \cdot J = \prod_\sigma (\sigma(J))^{n_\sigma}$. Note it is not obvious that the coefficients of θ are integral; this is a result of Deligne–Ribet.

It turns out that the above observation has a converse:

Proposition 3.2. [Tat84, Proposition 6.3] Consider the data $(K/F, S, v, w)$ as in the notation of Conjecture 2.1, where v is a *finite* place of F and $w = \mathfrak{P}$. Let $T \coloneqq S - \{v\}$, and suppose there exists an element $\alpha \in U_{v,S}(K)$ such that α generates $\theta_T \cdot \mathfrak{P}$ and $K(\alpha^{1/e_K})/F$ is an abelian extension. Then Conjecture 2.1 is true for the data $(K/F, S, v, w)$.

We now apply these facts to our situation with $F = \mathbf{Q}, K = \mathbf{Q}(\zeta_m)$, etc. First, $K(\beta(\mathfrak{p})^{1/e_K})$ is contained in $\mathbf{Q}(\zeta_{pm})$, meaning it is abelian over $\mathbf{Q}$. Moreover, by (a variant of) Stickelberger's Theorem as stated in [Coa77, Proposition 3.8*]:

Theorem 3.3. $\beta(\mathfrak{p}) \in \mathcal{O}_K$, and the factorization of the ideal generated by $\beta(\mathfrak{p})$ in $\mathcal{O}_K$ is $\theta_T \cdot \mathfrak{p}$.

In particular $\beta(\mathfrak{p})$ is a $\{p\}$-unit of K and Proposition 3.2 applies. □

For completeness we record the actual statement of the Brumer-Stark(-Tate) conjecture:

Conjecture 3.4 (Brumer-Stark(-Tate))**.** [Tat84, Conjecture 6.2] Let K/F be a finite abelian extension of number fields, and let $T \subseteq M_F$ be a finite subset of places containing the infinite ones and the ones that ramify in K. Define

$$K_T = \begin{cases} \{x \in K^* : |x|_w = 1 \text{ for all places } w \in M_K \text{ extending a place in } T\} & |T| \geq 2 \\ \{x \in K^* : |x|_w \text{ is constant over all places } w \in M_K \text{ extending the place in } T\} & |T| = 1. \end{cases}$$

Then for all fractional ideals I of O_K, there exists an element $\alpha \in K_T$ such that α generates $\theta_T \cdot I$ (where θ_T is defined as in Equation 3.7) and $K(\alpha^{1/e_K})/F$ is abelian.

One can view this as a generalization of Conjecture 2.1 that does not rely on the choice of the totally split place $v \in S$ as needed in that conjecture. More detailed discussions on the Brumer-Stark(-Tate) conjecture can be found at [DG11, Section 1.5] and [Tat84, IV.6].

To end this section we should remark that much more is known; for instance Stark originally proved Conjecture 2.1 when F is an imaginary quadratic field [Sta80]. This has the following interesting corollary:

Corollary 3.5. The Stark conjecture is true when $|S| = 2$.

Proof. S by assumption contains the infinite places, and unless F is $\mathbf{Q}$ or an imaginary quadratic field, in which case we are done, both $v, v' \in S$ are infinite. Again by the conditions on S, K/F is unramified at all finite places and v totally splits in K/F.

We claim v' is also totally split (i.e. unramified), so we may assume v' is a real place. For this, we may reduce to the case that K is the ray class field modulo v', so there is an isomorphism $I(\mathcal{O}_F)/i(F_{v',1}) \cong \mathrm{Gal}(K/F)$.[4] But $F_{v',1}$ is simply the usual subgroup of principal fractional ideals (there is only one real place to care about, and we can just take negatives to get a positive element), so class field theory tells us that K is the Hilbert class field of F.

The upshot is that S contains two totally split places, so Proposition 2.4 applies. □

4 Consequences

Here is an interesting consequence of the Stark conjecture in the $F = \mathbf{Q}$ case, which is mentioned in [Dum11]. Take $F = \mathbf{Q}$, $K = \mathbf{Q}(\sqrt{p})$ considered as a subfield of $\mathbf{R}$, where p is a prime equivalent to 1 mod 4, $S = \{p, \infty\}$, and $v = \infty$. Let w be the place K above v corresponding to the given (identity) embedding into $\mathbf{R}$, let w' be the other place above v, and let ϵ be a fundamental unit of K with $\epsilon > 1$. We know the Stark conjecture is true in this case but we did not explicitly compute the predicted unit u (which we may assume satisfies $u > 1$). On the other hand, Condition 2.2 gives, for $\mathbf{1}$ the trivial character of $G = \{1, \sigma\} \cong \mathbf{Z}/2\mathbf{Z}$ and χ the nontrivial character,

$$-\frac{1}{2}\log(p) = \zeta'_{\mathbf{Q},S}(0) = L'_S(\mathbf{1}, 0) = -\frac{1}{2}\left(\log(u) + \log(\sigma(u))\right),$$

since p is an S-regulator, and

$$h_K \log(\epsilon) = L'_S(\chi, 0) = -\frac{1}{2}\left(\log(u) - \log(\sigma(u))\right).$$

[4] We use Milne's notation here for $i(F_{v',1})$.

The first equality in the above line holds because Equation 1.1 gives $L_S(\chi, s) = \zeta_{K,S_K}(s)/\zeta_{\mathbf{Q},S}(s)$, and Theorem 1.3 tells us that the Taylor expansion of $L_S(\chi, s)$ begins in degree 1 with coefficient $\frac{h_{S_K} R_{S_K}}{h_S R_S}$ ($e_K = e_{\mathbf{Q}} = 2$). We visibly have $h_S = 1, R_S = \log(p)$, $h_{S_K} = h_K$, and $R_{S_K} = \log(\epsilon)\log(p)$.

This implies $u = \sqrt{p}\epsilon^{-h_K} > 0$. But because $K(\sqrt{u})$ is actually Galois over $\mathbf{Q}$, we need $|u|_{w'} = \sigma(u) > 0$ as well, as otherwise $\sqrt{\sigma(u)} \notin K(\sqrt{u})$ (this has at least one real embedding and is Galois over $\mathbf{Q}$, so must be totally real), contradiction. Hence $N_{K/\mathbf{Q}}(u) = -p/N_{K/\mathbf{Q}}(\epsilon)^{h_K}$ is positive, and so we have proved the following well-known result in a slightly different manner:[5]

Proposition 4.1. In the above situation, h_K is odd and the norm $N_{K/\mathbf{Q}}(\epsilon)$ of a fundamental unit must be -1.

One more application of the Stark conjecture is towards the explicit construction of class fields—in fact this was Stark's original motivation. Detailed examples, including the computation of the "Stark unit," are given in [Tat84, IV.4] and [DST97] (the latter in the case of cubic base fields). Here we will only vaguely summarize the basic idea. Let F be a totally real number field and H the Hilbert class field of F. If we take a small extension K of H (e.g. by adjoining a square root of some suitable element of the ground field F) so that K/F remains abelian and the hypotheses of Stark's conjecture are satisfied for $S = \{\text{the infinite places of } F\}$ so that the conjecture remains nontrivial (i.e. all but one of the infinite places in S ramify), then we can explicitly determine/approximate the Stark unit u via L-series computations and Criterion 2.2. In this way one can hopefully compute explicit generator(s) for K/F and ultimately H/F (e.g. by taking traces of generators of K). This is approximately the method taken in [DST97].

[5] I believe the usual proof that h_K is odd goes through the genus theory of number fields, but I am not familiar with this.

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